

Chow Forms, Chow Quotients and Quivers with Superpotential

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Abstract

We recast the correspondence between 3-dimensional toric Calabi-Yau singularities and quivers with superpotential in the setting of an abelian algebraic group $\mathcal{G}_{\mathbb{L}}$ acting on a linear space \mathbb{C}^N . We show how the quiver with superpotential gives a simple explicit description of the Chow forms of the closures of the orbits in the projective space \mathbb{P}^{N-1} . The resulting model of the orbit space is known as the Chow quotient of \mathbb{P}^{N-1} by $\mathcal{G}_{\mathbb{L}}$. As a by-product we prove a result about the relation between the quiver with superpotential and the principal \mathcal{A} -determinant in Gelfand-Kapranov-Zelevinsky's theory of discriminants.

1 Introduction.

One of the important themes in the physics literature on AdS/CFT is a correspondence between complex 3-dimensional Calabi-Yau singularities and quiver gauge theories. A breakthrough in this field was the discovery of an algorithm that realizes the correspondence for toric CY3 singularities [5, 7]. Experience with, for instance, the Gelfand-Kapranov-Zelevinsky theory of hypergeometric systems or De Bruijn's construction of Penrose tilings shows that sometimes great simplifications and new insights can be achieved by passing to an equivariant setting in higher dimensions. In [8] we adapted the aforementioned algorithm to a higher dimensional viewpoint. In the present paper we clarify some equivariance aspects. We study an $(N-2)$ -dimensional algebraic subgroup $\mathcal{G}_{\mathbb{L}}$ of \mathbb{C}^{*N} acting by diagonal matrices on \mathbb{C}^N and the induced action of a subgroup $\tilde{\mathcal{G}}_{\mathbb{L}}$ of $\mathcal{G}_{\mathbb{L}}$ on \mathbb{C}^N and of a quotientgroup $\overline{\mathcal{G}}_{\mathbb{L}}$ of $\mathcal{G}_{\mathbb{L}}$ on the projective space \mathbb{P}^{N-1} . Both groups $\tilde{\mathcal{G}}_{\mathbb{L}}$ and $\overline{\mathcal{G}}_{\mathbb{L}}$ have dimension $N-3$. We make models of the orbit spaces by embedding $\mathbb{C}^{*N}/\tilde{\mathcal{G}}_{\mathbb{L}}$ into some linear space and $\mathbb{C}^{*N}/\mathcal{G}_{\mathbb{L}}$ into some projective space and taking closures. We construct these embeddings directly from the quiver with superpotential \mathbf{P}

on the other side of the correspondence. For this we reformulate the data of \mathbf{P} in what we call the bi-adjacency matrix $\mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u})$; see Definition 1. Here \mathbf{z} denotes a tuple of variables with for every quiver arrow e one variable z_e and \mathbf{u} denotes a tuple of variables with for every quiver node i one variable u_i . The entries of $\mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u})$ are elements in the polynomial ring $\mathbb{Z}[\mathbf{z}, \mathbf{u}]$. Viewing u_1, \dots, u_N as coordinates on \mathbb{C}^N we get an action of the group $\mathcal{G}_{\mathbb{L}}$ on the polynomial ring $\mathbb{C}[\mathbf{u}] = \mathbb{C}[u_1, \dots, u_N]$. We show in Theorem 1 that under suitable conditions on \mathbf{P} and $\mathcal{G}_{\mathbb{L}}$ there is a character χ of $\mathcal{G}_{\mathbb{L}}$ such that in the polynomial ring $\mathbb{C}[\mathbf{z}, \mathbf{u}]$

$$\det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \xi \mathbf{u}) = \chi(\xi) \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}), \quad \forall \xi \in \mathcal{G}_{\mathbb{L}}.$$

The above mentioned group $\tilde{\mathcal{G}}_{\mathbb{L}}$ is the kernel of χ . By evaluating $\mathbf{u} = (u_1, \dots, u_N)$ at the points of \mathbb{C}^{*N} we obtain well defined maps

$$\mathbb{C}^{*N}/\tilde{\mathcal{G}}_{\mathbb{L}} \longrightarrow \mathbb{C}[\mathbf{z}]^{(\nu)}, \quad \mathbf{u} \mapsto \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}), \quad (1)$$

$$\mathbb{C}^{*N}/\mathcal{G}_{\mathbb{L}} \longrightarrow \mathbb{P}(\mathbb{C}[\mathbf{z}]^{(\nu)}), \quad \mathbf{u} \mapsto \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}) \bmod \mathbb{C}^*; \quad (2)$$

here $\mathbb{C}[\mathbf{z}]^{(\nu)}$ is the homogeneous part of degree $\nu = \deg_{\mathbf{z}} \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u})$ in the polynomial ring $\mathbb{C}[\mathbf{z}]$ and $\mathbb{P}(\mathbb{C}[\mathbf{z}]^{(\nu)})$ is the corresponding projective space. Taking Zariski closures of the images of the maps (1) and (2) one obtains models for the orbit spaces of $\tilde{\mathcal{G}}_{\mathbb{L}}$ acting on \mathbb{C}^N and $\mathcal{G}_{\mathbb{L}}$ acting on \mathbb{P}^{N-1} together with embeddings into $\mathbb{C}[\mathbf{z}]^{(\nu)}$ and $\mathbb{P}(\mathbb{C}[\mathbf{z}]^{(\nu)})$, respectively.

In Definition 3 we define the *complementary bi-adjacency matrix* $\mathbb{K}_{\mathbf{P}}^c(\mathbf{y}, \mathbf{u})$ and in Formula (21) we define a ring homomorphism \mathbf{y} from the polynomial ring $\mathbb{Z}[\mathbf{z}]$ to the ring $\mathcal{R}_{2,N}$ of homogeneous coordinates on the Grassmannian $\mathbb{G}(2, N)$. In Theorem 2 we show that for fixed $\mathbf{u} \in \mathbb{C}^{*N}$ the element $\det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ of $\mathcal{R}_{2,N}$ is a *Chow form* for the orbit closure $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ in \mathbb{P}^{N-1} . The maps (1) and (2) have obvious analogues:

$$\mathbb{C}^{*N}/\tilde{\mathcal{G}}_{\mathbb{L}} \longrightarrow \mathcal{R}_{2,N}^{(\nu)}, \quad \mathbf{u} \mapsto \det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u}), \quad (3)$$

$$\mathbb{C}^{*N}/\mathcal{G}_{\mathbb{L}} \longrightarrow \mathbb{P}(\mathcal{R}_{2,N}^{(\nu)}), \quad \mathbf{u} \mapsto \det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u}) \bmod \mathbb{C}^*; \quad (4)$$

The closure of the image of the map (4) in $\mathbb{P}(\mathcal{R}_{2,N}^{(\nu)})$ is a model for the orbit space of $\mathcal{G}_{\mathbb{L}}$ acting on \mathbb{P}^{N-1} known as the *Chow quotient*; see [6]. In [6] one can also find a discussion of how this model compares with GIT quotients.

A priori $\det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ is an element in $\mathcal{R}_{2,N}[\mathbf{u}] = \mathcal{R}_{2,N}[u_1, \dots, u_N]$, i.e. in the ring of homogeneous coordinates on $\mathbb{G}(2, N) \times \mathbb{P}^{N-1}$. In (4) this element is evaluated at points of \mathbb{P}^{N-1} . One can also evaluate it at points of $\mathbb{G}(2, N)$ and find polynomials in the ring $\mathbb{C}[\mathbf{u}] = \mathbb{C}[u_1, \dots, u_N]$. A point of $\mathbb{G}(2, N)$ is a line in \mathbb{P}^{N-1} and the corresponding polynomial is the equation for the image of this line in the Chow quotient for $\mathcal{G}_{\mathbb{L}}$ acting on \mathbb{P}^{N-1} .

The \mathbb{L} which appears in the above presentation as a subscript is in fact the character lattice of the group $\mathbb{C}^{*N}/\mathcal{G}_{\mathbb{L}}$. It is a rank 2 subgroup of \mathbb{Z}^N and thus defines a point \mathfrak{l} in the Grassmannian $\mathbb{G}(2, N)$. In Theorem 3 we show that evaluating $\det \mathbb{K}_{\mathfrak{p}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ at the point \mathfrak{l} yields the principal \mathcal{A} -determinant of Gelfand-Kapranov-Zelevinsky [4], if $\mathcal{G}_{\mathbb{L}}$ is connected. In [8] this result was observed in some examples and then conjectured to hold in general.

In Figures 1, 2, 3 and Section 7 we give one example of the theory presented in this paper. Some further examples can be found in [8].

2 The lattice \mathbb{L} and associated groups.

Throughout this paper \mathbb{L} is a rank 2 subgroup of \mathbb{Z}^N not contained in any of the standard coordinate hyperplanes and contained in the kernel of the map

$$h : \mathbb{Z}^N \longrightarrow \mathbb{Z}, \quad h(z_1, \dots, z_N) = z_1 + \dots + z_N. \quad (5)$$

Unlike what seems to be practice in the theory of Gelfand, Kapranov and Zelevinsky [4], we do allow torsion in the quotient group \mathbb{Z}^N/\mathbb{L} . Denoting by $\mathbf{e}_1, \dots, \mathbf{e}_N$ the standard basis of \mathbb{Z}^N we set

$$\mathcal{G}_{\mathbb{L}} := \text{Hom}(\mathbb{Z}^N/\mathbb{L}, \mathbb{C}^*), \quad \mathbf{a}_i := \mathbf{e}_i \bmod \mathbb{L} \text{ in } \mathbb{Z}^N/\mathbb{L}.$$

$\mathcal{G}_{\mathbb{L}}$ is an algebraic group for which the connected component containing the identity is isomorphic to the complex torus $(\mathbb{C}^*)^{N-2}$ and the group of connected components is isomorphic to the finite abelian group $(\mathbb{Z}^N/\mathbb{L})_{\text{tors}}$. The inclusion $\mathbb{L} \hookrightarrow \mathbb{Z}^N$ induces the inclusion of groups

$$\mathcal{G}_{\mathbb{L}} \hookrightarrow \mathbb{C}^{*N}, \quad \xi \mapsto (\xi(\mathbf{a}_1), \xi(\mathbf{a}_2), \dots, \xi(\mathbf{a}_N))$$

and thus an action of $\mathcal{G}_{\mathbb{L}}$ on \mathbb{C}^N and \mathbb{P}^{N-1} . The map h induces a homomorphism $\bar{h} : \mathbb{Z}^N/\mathbb{L} \rightarrow \mathbb{Z}$ such that $\bar{h}(\mathbf{a}_i) = 1$ for all i . As a consequence there is a subgroup $\mathcal{G}'_{\mathbb{L}}$ of $\mathcal{G}_{\mathbb{L}}$ consisting of those $\xi \in \mathcal{G}_{\mathbb{L}}$ for which $\xi(\mathbf{a}_1) = \dots = \xi(\mathbf{a}_N)$. So, in fact $\mathcal{G}_{\mathbb{L}}$ acts on \mathbb{P}^{N-1} via the quotient group $\bar{\mathcal{G}}_{\mathbb{L}} := \mathcal{G}_{\mathbb{L}}/\mathcal{G}'_{\mathbb{L}}$.

The subgroup $\tilde{\mathcal{G}}_{\mathbb{L}}$ of $\mathcal{G}_{\mathbb{L}}$ is the kernel of some character χ of $\mathcal{G}_{\mathbb{L}}$. The construction is slightly involved. We choose a basis for \mathbb{L} and represent the inclusion $\mathbb{L} \hookrightarrow \mathbb{Z}^N$ by a matrix $B = (b_{ij})_{i=1,2;j=1,\dots,N}$ with entries in \mathbb{Z} . Let $\beta_1, \dots, \beta_N \in \mathbb{Z}^2$ denote the columns of B . The complete fan in \mathbb{R}^2 with 1-dimensional rays $\mathbb{R}_{\geq 0}\beta_i$ ($i = 1, \dots, N$) is the so-called *secondary fan* associated with \mathbb{L} ; see Figure 1 for an example. Take $\mathbf{c} \in \mathbb{R}^2 \setminus \bigcup_{i=1}^N \mathbb{R}_{\geq 0}\beta_i$ and set $L_{\mathbf{c}} = \{ \{i, j\} \subset \{1, \dots, N\} \mid \mathbf{c} \in \mathbb{R}_{\geq 0}\beta_i + \mathbb{R}_{\geq 0}\beta_j \}$. We now define:

$$\mathbf{a}_0 := \sum_{\{i,j\} \in L_{\mathbf{c}}} |\det(\beta_i, \beta_j)| (\mathbf{a}_i + \mathbf{a}_j), \quad (6)$$

$$\chi : \mathcal{G}_{\mathbb{L}} \rightarrow \mathbb{C}^*, \quad \chi(\xi) := \xi(\mathbf{a}_0), \quad (7)$$

$$\tilde{\mathcal{G}}_{\mathbb{L}} := \ker \chi. \quad (8)$$

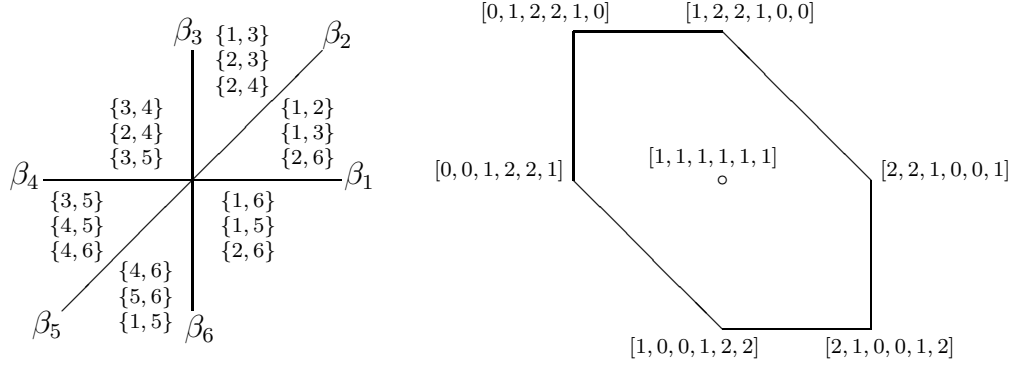


Figure 1: *Secondary fan with lists $L_{\mathbf{c}}$ (left) and secondary polygon with vectors $\sum_{\{i,j\} \in L_{\mathbf{c}}} (\mathbf{e}_i + \mathbf{e}_j)$ (right) for $\mathbb{L} = \mathbb{Z}^2 \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$. This shows $\mathbf{a}_0 = (1, 1, 1, 1, 1, 1) \bmod \mathbb{L}$.*

It follows from [8] Eqs.(6), (18) that \mathbf{a}_0 does not depend on the particular choice of the vector \mathbf{c} . Changing the basis for \mathbb{L} multiplies B with a 2×2 -matrix with determinant ± 1 . So, \mathbf{a}_0 does also not depend on the particular choice of the basis for \mathbb{L} .

3 The quiver with superpotential P .

On the quiver side we look at a quiver with a special kind of superpotential $P = (P^0, P^1, P^{2\bullet}, P^{2\circ}, s, t, \mathbf{b}, \mathbf{w})$. It consists of four finite sets $P^0, P^1, P^{2\bullet}, P^{2\circ}$ and four maps

$$s, t : P^1 \rightarrow P^0, \quad \mathbf{b} : P^1 \rightarrow P^{2\bullet}, \quad \mathbf{w} : P^1 \rightarrow P^{2\circ}. \quad (9)$$

The elements of P^0 are called 0-cells or nodes or vertices. The elements of P^1 are called 1-cells or arrows or edges. The elements of $P^{2\bullet}$ (resp. $P^{2\circ}$) are called black (resp. white) 2-cells. An arrow e is oriented from $s(e)$ to $t(e)$.

Througout this paper the following condition is assumed to be satisfied:

Condition 1 .

1. *The directed graph (P^0, P^1, s, t) is connected and has no oriented cycles of length ≤ 2 and in every node $v \in P^0$ there are as many incoming as outgoing arrows; i.e. $\#\{e \in P^1 | t(e) = v\} = \#\{e \in P^1 | s(e) = v\}$.*
2. *There are as many black as white cells; i.e. $\#P^{2\bullet} = \#P^{2\circ}$.*
3. *For every $\mathbf{b} \in P^{2\bullet}$ and every $\mathbf{w} \in P^{2\circ}$ the sets $\{e \in P^1 | \mathbf{b}(e) = \mathbf{b}\}$ and $\{e \in P^1 | \mathbf{w}(e) = \mathbf{w}\}$ are connected oriented cycles; by this we mean that the elements can be ordered (e_1, e_2, \dots, e_r) such that $t(e_i) = s(e_{i+1})$ for $i = 1, \dots, r-1$ and $t(e_r) = s(e_1)$.*



Figure 2: *On the left: the quiver with superpotential P ; opposite sides of the hexagon must be identified; P actually lies on a torus. On the right: part of the lift of P on the universal cover of the torus.*

One can realize P geometrically as an oriented surface without boundary by representing every black 2-cell \mathbf{b} (resp. white 2-cell \mathbf{w}) as a convex polygon with sides labeled by the elements $e \in P^1$ with $\mathbf{b}(e) = \mathbf{b}$ (resp. $\mathbf{w}(e) = \mathbf{w}$) and by gluing these polygons along sides with the same label. In quiver theory one views (P^0, P^1, s, t) as a quiver (=directed graph) and the boundary cycles of the 2-cells as the terms of a superpotential; cf. [8] 8.6.

The structure of P described in (9) can be accurately captured in its bi-adjacency matrix:

Definition 1 *The bi-adjacency matrix of P is the square matrix $\mathbb{K}_P(\mathbf{z}, \mathbf{u})$ with rows (resp. columns) corresponding with the black (resp. white) 2-cells, with entries in the polynomial ring $\mathbb{Z}[z_e, u_i \mid e \in P^1, i \in P^0]$ and (\mathbf{b}, \mathbf{w}) -entry:*

$$(\mathbb{K}_P(\mathbf{z}, \mathbf{u}))_{\mathbf{b}, \mathbf{w}} = \sum_{e \in P^1: \mathbf{b}(e)=\mathbf{b}, \mathbf{w}(e)=\mathbf{w}} z_e u_{s(e)} u_{t(e)}. \quad (10)$$

This definition of the bi-adjacency matrix is the same as [8] Eq.(1), except that in loc. cit. we used a \mathbb{C} -valued function ϖ on P^1 instead of the formal variables z_e . The products $u_{s(e)} u_{t(e)}$ of commuting variables do not show the orientation of the edge e . As explained in [8] §8.10, the orientation can easily be reconstructed. So there is no loss of information in going from (9) to (10). The bi-adjacency matrix $\mathbb{K}_P(\mathbf{z}, \mathbf{u})$ of P is also the *Kasteleyn matrix of a twist of P* ; see [8] Section 8.

4 Compatibility of \mathbb{L} and P .

In [8] §6 we described an algorithm, based on the *Fast Inverse Algorithm* of [5] and the *untwisting procedure* of [2], for constructing from a rank 2 lattice \mathbb{L} as in Section 2 a quiver with superpotential P as in Section 3, such that also Condition 2 (below) is satisfied. We isolated Conditions 1 and 2 because this

$\varepsilon_{\mathbf{b}} \downarrow : \varepsilon_{\mathbf{w}} \rightarrow$	$\parallel (0, -1, -1, 0, 0, 0) \parallel$	$\parallel (-1, -1, 0, 0, 0, 0) \parallel$	$\parallel (0, 0, -1, -1, 0, 0) \parallel$
$(0, 0, 0, 0, 0, 0)$	$z_1 u_2 u_3 + z_2 u_5 u_6$	$z_3 u_1 u_2 + z_4 u_4 u_5$	$z_5 u_3 u_4 + z_6 u_1 u_6$
$(0, 0, -1, 0, 0, 1)$	$z_7 u_2 u_6$	$z_8 u_2 u_4$	$z_9 u_4 u_6$
$(0, -1, 0, 0, 1, 0)$	$z_{10} u_3 u_5$	$z_{11} u_1 u_5$	$z_{12} u_1 u_3$

Figure 3: The bi-adjacency matrix for \mathbf{P} as in Figure 2 and on its sides vectors (to be read modulo the \mathbb{L} of Figure 1) which verify Condition 2.

is exactly what we need in the present paper and because these conditions can be checked without worrying about the algorithm of [8].

Condition 2 For every $\mathbf{b} \in \mathbf{P}^{2\bullet}$ and $\mathbf{w} \in \mathbf{P}^{2\circ}$ there are elements $\varepsilon_{\mathbf{b}}$ and $\varepsilon_{\mathbf{w}}$ in \mathbb{Z}^N/\mathbb{L} such that for every $e \in \mathbf{P}^1$ and with \mathbf{a}_0 as in (6) one has

$$\mathbf{a}_{s(e)} + \mathbf{a}_{t(e)} = \varepsilon_{\mathbf{b}(e)} - \varepsilon_{\mathbf{w}(e)}, \quad \mathbf{a}_0 = \sum_{\mathbf{b} \in \mathbf{P}^{2\bullet}} \varepsilon_{\mathbf{b}} - \sum_{\mathbf{w} \in \mathbf{P}^{2\circ}} \varepsilon_{\mathbf{w}}. \quad (11)$$

The ring $\mathbb{C}[u_i \mid i \in \mathbf{P}^0] = \mathbb{C}[u_1, \dots, u_N]$ is the coordinate ring of \mathbb{C}^N and therefore carries an action of the group $\mathcal{G}_{\mathbb{L}}$:

$$\xi u_i := \xi(\mathbf{a}_i) u_i \quad \text{for } i = 1, \dots, N, \xi \in \mathcal{G}_{\mathbb{L}}.$$

Theorem 1 Assume \mathbb{L} and \mathbf{P} satisfy Conditions 1 and 2. Then we have for every $\xi \in \mathcal{G}_{\mathbb{L}}$:

$$\mathbb{K}_{\mathbf{P}}(\mathbf{z}, \xi \mathbf{u}) = \text{diag}(\xi(\varepsilon_{\mathbf{b}})) \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}) \text{diag}(\xi(-\varepsilon_{\mathbf{w}})), \quad (12)$$

$$\det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \xi \mathbf{u}) = \chi(\xi) \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}), \quad (13)$$

$$\deg_{\mathbf{z}} \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}) = \frac{1}{2} \deg_{\mathbf{u}} \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}) = \sharp \mathbf{P}^{2\bullet} = \sharp \mathbf{P}^{2\circ} = \frac{1}{2} \bar{h}(\mathbf{a}_0). \quad (14)$$

with character χ as in (7) and the entries of the diagonal matrices $\text{diag}(\xi(\varepsilon_{\mathbf{b}}))$ and $\text{diag}(\xi(-\varepsilon_{\mathbf{w}}))$ labeled with the black and white 2-cells in agreement with the labeling of the rows and columns of $\mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u})$.

Proof. (12), (13) and the first three equalities in (14) are obvious. The first half of (11) implies that there is a $k \in \mathbb{Z}$ such that $\bar{h}(\varepsilon_{\mathbf{b}}) = \bar{h}(\varepsilon_{\mathbf{w}}) + 2 = k$ for all $\mathbf{b} \in \mathbf{P}^{2\bullet}$ and $\mathbf{w} \in \mathbf{P}^{2\circ}$. The last equality in (14) now follows from the second half of (11). \blacksquare

5 Lines in projective space and Chow forms.

Every line in \mathbb{P}^{N-1} is of the form

$$\mathcal{L}_Y := \{[u_1 : \dots : u_N] \in \mathbb{P}^{N-1} \mid u_j = t_1 y_{1j} + t_2 y_{2j}, [t_1 : t_2] \in \mathbb{P}^1\}$$

with $Y = (y_{ij})_{i=1,2;j=1,\dots,N}$ a $2 \times N$ complex matrix of rank 2. Let $M(2, N)$ denote the set of $2 \times N$ complex matrices of rank 2. As $\mathcal{L}_{Y'} = \mathcal{L}_Y$ if and only if there is an invertible 2×2 -matrix g such that $Y' = gY$, the lines in \mathbb{P}^{N-1} correspond bijectively with the points of the Grassmannian

$$\mathbb{G}(2, N) := Gl(2) \backslash M(2, N).$$

Natural homogeneous coordinates on $\mathbb{G}(2, N)$ are the *Plücker coordinates* Y_{km} of the matrix $Y = (y_{ij})_{i=1,2;j=1,\dots,N}$:

$$Y_{km} := y_{1k}y_{2m} - y_{2k}y_{1m}, \quad k, m = 1, \dots, N. \quad (15)$$

In more algebraic terms (see [4] p.96), *the ring of homogeneous coordinates on $\mathbb{G}(2, N)$ in the Plücker embedding* is the ring

$$\mathcal{R}_{2,N} := \mathbb{C}[\Upsilon_{km} \mid k, m = 1, \dots, N] / \mathcal{I}, \quad (16)$$

where \mathcal{I} is the ideal in the polynomial ring $\mathbb{C}[\Upsilon_{km}]$ generated by the elements

$$\Upsilon_{km} + \Upsilon_{mk} \quad \text{and} \quad \Upsilon_{ij}\Upsilon_{km} + \Upsilon_{ik}\Upsilon_{mj} + \Upsilon_{im}\Upsilon_{jk}, \quad \forall i, j, k, m. \quad (17)$$

Y_{km} is then the element $\Upsilon_{km} \bmod \mathcal{I}$ of $\mathcal{R}_{2,N}$.

Definition 2 (see [4] pp.99,100,123) *Let \mathcal{X} be an irreducible subvariety of \mathbb{P}^{N-1} of codimension 2 and degree d . Then the set of lines \mathcal{L}_Y which intersect \mathcal{X} is an irreducible hypersurface $\mathcal{Z}(\mathcal{X})$ in the Grassmannian $\mathbb{G}(2, N)$, called the associated hypersurface of \mathcal{X} . The hypersurface $\mathcal{Z}(\mathcal{X})$ is given by the vanishing of a single, up to a multiplicative constant unique, irreducible homogeneous element of degree d in $\mathcal{R}_{2,N}$, the so-called Chow form of \mathcal{X} . The Chow form of a codimension 2 subvariety of \mathbb{P}^{N-1} is the product of the Chow forms of its irreducible components.*

Our aim is to explicitly compute the Chow form of the orbit closure $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ when $[\mathbf{u}] \in \mathbb{P}^{N-1}$ is a point with all homogeneous coordinates $\neq 0$.

Proposition 1 *For $\mathbf{u} = (u_1, \dots, u_N) \in \mathbb{C}^{*N}$ we denote its inverse in \mathbb{C}^{*N} by $\mathbf{u}^{-1} = (u_1^{-1}, \dots, u_N^{-1})$ and its image in \mathbb{P}^{N-1} by $[\mathbf{u}] = [u_1 : \dots : u_N]$. For $Y \in M(2, N)$ we denote by \mathbf{Y}_{st} the tuple of complex numbers $(Y_{s(e)t(e)})_{e \in \mathbb{P}^1}$, where Y_{km} is the Plücker coordinate as in (15). Then*

$$\text{point } [\mathbf{u}] \text{ lies on line } \mathcal{L}_Y \quad \Rightarrow \quad \det \mathbb{K}_{\mathbb{P}}(\mathbf{Y}_{st}, \mathbf{u}^{-1}) = 0. \quad (18)$$

Proof. From the fact that a $3 \times N$ -matrix has $\text{rank} \leq 2$ if and only if all its 3×3 -subdeterminants vanish, we see that *point* $[\mathbf{u}]$ *lies on line* \mathcal{L}_Y if and only if

$$u_k^{-1}u_m^{-1}Y_{km} + u_m^{-1}u_j^{-1}Y_{mj} + u_j^{-1}u_k^{-1}Y_{jk} = 0 \quad (19)$$

for every triple of elements j, k, m in $\{1, \dots, N\}$. Adding equations from the system (19) yields an equation

$$\sum_{i=1}^r Y_{k_i k_{i+1}} u_{k_i}^{-1} u_{k_{i+1}}^{-1} = 0 \quad (20)$$

for every cycle (k_1, \dots, k_r) , $k_{r+1} = k_1$, of distinct elements of $\{1, \dots, N\}$. In particular, there is such an equation (20) for the cycle of vertices of a black 2-cell of \mathbf{P} and the terms of that equation match bijectively with the adjacent white 2-cells. This means that in every row of the matrix $\mathbb{K}_{\mathbf{P}}(\mathbf{Y}_{st}, \mathbf{u}^{-1})$ the sum of the entries is 0. Consequently $\det \mathbb{K}_{\mathbf{P}}(\mathbf{Y}_{st}, \mathbf{u}^{-1}) = 0$. \blacksquare

An immediate consequence of Proposition 1 and Theorem 1 is:

Corollary 1 *Assume \mathbb{L} and \mathbf{P} satisfy Conditions 1 and 2. Let \mathbf{u} and Y be as in Proposition 1. Then $\det \mathbb{K}_{\mathbf{P}}(\mathbf{Y}_{st}, \mathbf{u}^{-1}) = 0$ if the line \mathcal{L}_Y intersects the closure $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ of the orbit $\mathcal{G}_{\mathbb{L}}[\mathbf{u}]$ in \mathbb{P}^{N-1} .*

Proof. Indeed (13) and (18) imply $\det \mathbb{K}_{\mathbf{P}}(\mathbf{Y}_{st}, \mathbf{u}^{-1}) = 0$ if the line \mathcal{L}_Y intersects the orbit $\mathcal{G}_{\mathbb{L}}[\mathbf{u}]$. The corollary follows because the set of lines which intersect $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ is the closure in the Grassmannian $\mathbb{G}(2, N)$ of the set of lines which intersect $\mathcal{G}_{\mathbb{L}}[\mathbf{u}]$. \blacksquare

In order to remedy for the appearing inverses we now introduce the complementary bi-adjacency matrix:

Definition 3 *The complementary bi-adjacency matrix $\mathbb{K}_{\mathbf{P}}^c(\mathbf{z}, \mathbf{u})$ is*

$$\mathbb{K}_{\mathbf{P}}^c(\mathbf{z}, \mathbf{u}) := u_1 \cdot \dots \cdot u_N \cdot \mathbb{K}_{\mathbf{P}}(\mathbf{z}, (u_1^{-1}, \dots, u_N^{-1})).$$

with, as usual, $\mathbf{u} = (u_1, \dots, u_N)$. The matrix $\mathbb{K}_{\mathbf{P}}^c(\mathbf{z}, \mathbf{u})$ has entries in the polynomial ring $\mathbb{Z}[\mathbf{z}, \mathbf{u}]$.

A further consequence of Proposition 1 and Theorem 1 is:

Corollary 2 *Assume that \mathbb{L} and \mathbf{P} satisfy Conditions 1 and 2. Let \mathbf{y} denote the ringhomomorphism (cf. (16))*

$$\mathbf{y} : \mathbb{Z}[z_e \mid e \in \mathbf{P}^1] \longrightarrow \mathcal{R}_{2,N}, \quad \mathbf{y}(z_e) = Y_{s(e)t(e)}. \quad (21)$$

*Then for $\mathbf{u} \in \mathbb{C}^{*N}$ the Chow form of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ divides $\det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ in $\mathcal{R}_{2,N}$.*

Proof. There is a slight subtlety because torsion in \mathbb{Z}^N/\mathbb{L} makes $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ reducible. If \mathcal{X} is an irreducible component of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ it follows from Corollary 1 that the Chow form of \mathcal{X} divides $\det \mathbb{K}_{\mathbb{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$, when both are viewed as elements of $\mathcal{R}_{2,N}$. Since an irreducible codimension 2 subvariety of \mathbb{P}^{N-1} is uniquely determined by its Chow form (see [4] p.102 Prop.2.5), the Chow forms of different irreducible components of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ are not equal. It follows that $\det \mathbb{K}_{\mathbb{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ is divisible by the Chow form of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$. \blacksquare

Theorem 2 *Assume that \mathbb{L} and \mathbb{P} satisfy Conditions 1 and 2. Let $\mathbf{u} \in \mathbb{C}^{*N}$. Then $\det \mathbb{K}_{\mathbb{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ is a Chow form for $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$.*

N.B. We write here a Chow form to emphasize that it is only determined up to a non-zero multiplicative constant.

Proof. In view of Corollary 2 we need only prove that the Chow form of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ and $\det \mathbb{K}_{\mathbb{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ are elements of $\mathcal{R}_{2,N}$ with the same degree. Take \mathbb{L}^0 such that $\mathbb{L} \subset \mathbb{L}^0 \subset \mathbb{Z}^N$ and $\mathbb{L}^0/\mathbb{L} = (\mathbb{Z}^N/\mathbb{L})_{\text{tors}}$. Set

$$\mathcal{G}_{\mathbb{L}}^0 := \text{Hom}(\mathbb{Z}^N/\mathbb{L}^0, \mathbb{C}^*), \quad \mathbf{a}_i^0 := \mathbf{e}_i \bmod \mathbb{L}^0, \quad \mathcal{A}^0 := \{\mathbf{a}_1^0, \dots, \mathbf{a}_N^0\}.$$

Then $\mathcal{G}_{\mathbb{L}}^0$ is the connected component containing the identity in $\mathcal{G}_{\mathbb{L}}$. Via the action of \mathbb{C}^{*N} every irreducible component of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ together with its embedding into \mathbb{P}^{N-1} is isomorphic to $\overline{\mathcal{G}_{\mathbb{L}}^0 \mathbf{1}} \hookrightarrow \mathbb{P}^{N-1}$, where $\mathbf{1} \in \mathbb{P}^{N-1}$ has all its components equal to 1. In [4] p.166 the toric variety $\overline{\mathcal{G}_{\mathbb{L}}^0 \mathbf{1}}$ is denoted as $X_{\mathcal{A}^0}$. Since \mathbb{L} is contained in the kernel of the map h (see (5)) the set \mathcal{A}^0 is contained in an affine hyperplane in $\mathbb{Z}^N/\mathbb{L}^0 = \mathbb{Z}^{N-2}$. By [4] p.203 Theorem 2.3 and p.99 Proposition 2.2 the degree of the toric variety $X_{\mathcal{A}^0} \subset \mathbb{P}^{N-1}$ and the degree of its Chow form in $\mathcal{R}_{2,N}$ are both equal to the volume $\text{vol}_{\mathcal{A}^0}$ of the convex hull of \mathcal{A}^0 , with the volume normalized so that the standard unit cube in \mathbb{Z}^{N-3} has volume $(N-3)!$ (see [4] p.182). Let $\mathbf{a}_0^0 \in \mathbb{Z}^N/\mathbb{L}^0$ be the image of $\mathbf{a}_0 \in \mathbb{Z}^N/\mathbb{L}$. It is well-known and easy to prove (see e.g. [8] Eq.(19)) that $\text{vol}_{\mathcal{A}^0} = \frac{1}{2}h(\mathbf{a}_0^0)$. On the other hand it is obvious that $h(\mathbf{a}_0)h(\mathbf{a}_0^0)^{-1} = \sharp(\mathbb{L}^0/\mathbb{L})$ and that this equals the number of irreducible components of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$. Thus the degree of the Chow form of $\overline{\mathcal{G}_{\mathbb{L}}[\mathbf{u}]}$ is equal to $\frac{1}{2}h(\mathbf{a}_0)$. It follows from (10) and (14) that this is also the degree of $\det \mathbb{K}_{\mathbb{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ as an element of $\mathcal{R}_{2,N}$. \blacksquare

6 The principal \mathcal{A} -determinant.

In this section we prove Conjecture 10.5 of [8] by combining Theorem 2 and [1] Proposition 3.2. The result is:

Theorem 3 Assume that Conditions 1 and 2 are satisfied and that \mathbb{Z}^N/\mathbb{L} has no torsion. Let $B = (b_{ij})_{i=1,2; j=1,\dots,N}$ be a $2 \times N$ -matrix with entries in \mathbb{Z} which gives the inclusion $\mathbb{L} \hookrightarrow \mathbb{Z}^N$ and let \mathbf{B}_{st} denote the ring homomorphism

$$\mathbf{B}_{st} : \mathbb{Z}[z_e \mid e \in \mathbf{P}^1] \longrightarrow \mathbb{Z}, \quad \mathbf{B}_{st}(z_e) = b_{1s(e)}b_{2t(e)} - b_{1t(e)}b_{2s(e)}.$$

Set $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ and let $E_{\mathcal{A}}(f)$ denote the principal \mathcal{A} -determinant defined in [4] p.297 Eq. (1.1) for the Laurent polynomial $f = \sum_{i=1}^N u_i \mathbf{x}^{\mathbf{a}_i}$.

Then the following equality holds in the polynomial ring $\mathbb{Z}[u_1, \dots, u_N]$

$$E_{\mathcal{A}}(f) = \pm \mathbf{B}_{st}(\det \mathbb{K}_{\mathbf{P}}^c(\mathbf{z}, \mathbf{u})) . \quad (22)$$

Proof. One can trace back the definition of the principal \mathcal{A} -determinant $E_{\mathcal{A}}(f)$ from [4] p.297 Eq.(1.1) to the definition of the Chow form of the subvariety $X_{\mathcal{A}} = \overline{\mathcal{G}_{\mathbb{L}}1}$ of \mathbb{P}^{N-1} on [4] p.100. That is also what Dickenstein and Sturmfels do before defining the principal \mathcal{A} -determinant, which they call the *full discriminant*, in [1] Definition 3.1. The result in [1] Eq. (3.2) is a more or less immediate consequence of this definition. The above Equation (22) now follows by substituting the expression for the Chow form given in Theorem 2 into [1] Eq. (3.2). ■

7 An example.

Figures 1, 2 and 3 show \mathbb{L} , \mathbf{P} , the bi-adjacency matrix and the verification of Condition 2 for the example which in the physics literature (e.g. [3]) is known as *model I of Del Pezzo 3*. The right-hand picture in Figure 2 is the dual of the *brane tiling* in Figure 1 of [3]. One should keep in mind that in [3] toric data for the singularity are extracted from the *Kasteleyn matrix* (see [3] Eqs. (3.4), (3.5)), whereas we work with the bi-adjacency matrix. In this particular example the Kasteleyn matrix and the bi-adjacency matrix give the same toric data, but in general they lead to different toric data.

In the context of hypergeometric systems this example is Appell's F_1 ; see [8] §4. The group \mathbb{Z}^N/\mathbb{L} is torsion free and can be identified with \mathbb{Z}^4 . This puts $\mathbf{a}_1, \dots, \mathbf{a}_6$ as points in a hyperplane in \mathbb{Z}^4 . Figure 4 shows the points $\mathbf{a}_1, \dots, \mathbf{a}_6$ and their convex hull (often called *the primary polytope*) situated in this 3-dimensional hyperplane. \mathbb{L} is the lattice of affine relations between these six points.

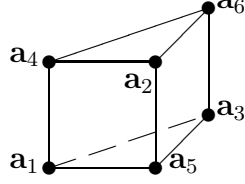


Figure 4: The set $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_6\}$ for $\mathbb{L} = \mathbb{Z}^2 \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$.

From the bi-adjacency matrix in Figure 3 one easily gets the determinants

$$\begin{aligned} \det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}) = & (z_2 z_8 z_{12} + z_3 z_9 z_{10} + z_5 z_7 z_{11} - z_6 z_8 z_{10} - z_1 z_9 z_{11} - z_4 z_7 z_{12}) \mathbf{u}^{[1,1,1,1,1,1]} \\ & + z_1 z_8 z_{12} \mathbf{u}^{[1,2,2,1,0,0]} + z_4 z_9 z_{10} \mathbf{u}^{[0,0,1,2,2,1]} + z_6 z_7 z_{11} \mathbf{u}^{[2,1,0,0,1,2]} \\ & - z_5 z_8 z_{10} \mathbf{u}^{[0,1,2,2,1,0]} - z_2 z_9 z_{11} \mathbf{u}^{[1,0,0,1,2,2]} - z_3 z_7 z_{12} \mathbf{u}^{[2,2,1,0,0,1]}. \end{aligned}$$

$$\begin{aligned} \det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u}) = & (\mathbf{Y}_{56|24|13} + \mathbf{Y}_{12|46|35} + \mathbf{Y}_{34|62|51} - \mathbf{Y}_{61|24|35} - \mathbf{Y}_{23|46|51} - \mathbf{Y}_{45|62|13}) \mathbf{u}^{[2,2,2,2,2,2]} \\ & + \mathbf{Y}_{23|24|13} \mathbf{u}^{[2,1,1,2,3,3]} + \mathbf{Y}_{45|46|35} \mathbf{u}^{[3,3,2,1,1,2]} + \mathbf{Y}_{61|62|51} \mathbf{u}^{[1,2,3,3,2,1]} \\ & - \mathbf{Y}_{34|24|35} \mathbf{u}^{[3,2,1,1,2,3]} - \mathbf{Y}_{56|46|51} \mathbf{u}^{[2,3,3,2,1,1]} - \mathbf{Y}_{12|62|13} \mathbf{u}^{[1,1,2,3,3,2]}. \end{aligned}$$

with notations $\mathbf{u}^{[a,b,c,d,e,f]} := u_1^a u_2^b u_3^c u_4^d u_5^e u_6^f$ and $\mathbf{Y}_{ab|cd|ef} := Y_{ab} Y_{cd} Y_{ef}$; here the Y_{km} are the Plücker coordinates on $\mathbb{G}(2, 6)$; see (15)-(17). These formulas yield for every $\mathbf{u} \in (\mathbb{C}^*)^6$ elements in $\mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \dots, z_{12}]$ and $\mathcal{R}_{2,6}$, respectively. Taking the closure of $\{\det \mathbb{K}_{\mathbf{P}}(\mathbf{z}, \mathbf{u}) \mid \mathbf{u} \in (\mathbb{C}^*)^6\}$ in $\mathbb{C}[\mathbf{z}]$ adds six lines which correspond to the six sides of the secondary polygon in Figure 1; the top side, for instance, gives the line $\{(z_1 z_8 z_{12} x_1 - z_5 z_8 z_{10} x_2) \mid [x_1 : x_2] \in \mathbb{P}^1\}$. A similar remark holds for the closure of $\{\det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u}) \mid \mathbf{u} \in (\mathbb{C}^*)^6\}$.

On the other hand evaluating $\det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ at the point \mathbf{l} amounts in this case to setting all $Y_{km} = 1$. This results in a polynomial in $\mathbb{C}[u_1, \dots, u_6]$:

$$\begin{aligned} \det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})|_{\text{at } \mathbf{l}} = & \mathbf{u}^{[2,1,1,2,3,3]} + \mathbf{u}^{[3,3,2,1,1,2]} + \mathbf{u}^{[1,2,3,3,2,1]} - \mathbf{u}^{[3,2,1,1,2,3]} - \mathbf{u}^{[2,3,3,2,1,1]} - \mathbf{u}^{[1,1,2,3,3,2]} \\ = & u_1 u_2 u_3 u_4 u_5 u_6 (u_1 u_2 - u_4 u_5) (u_3 u_4 - u_1 u_6) (u_5 u_6 - u_2 u_3). \end{aligned}$$

Note that in the first expression for $\det \mathbb{K}_{\mathbf{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})|_{\text{at } \mathbf{l}}$ all monomials are invariant for the action of the group $\tilde{\mathcal{G}}_{\mathbb{L}}$ on $\mathbb{C}[u_1, \dots, u_6]$. This invariance does not hold for the factors in the second expression. On the other hand, the factors in this factorization correspond with the faces of the primary polytope in Figure 4 in agreement with the prime factorization of the principal \mathcal{A} -determinant in [4] p.299 Theorem 1.2.

There can not be a similar factorization of $\det \mathbb{K}_{\mathbb{P}}^c(\mathbf{y}(\mathbf{z}), \mathbf{u})$ in the ring $\mathcal{R}_{2,6}[u_1, \dots, u_6]$, because that would upon specializing $u_1, \dots, u_6 \in \mathbb{C}^*$ give a factorization in $\mathcal{R}_{2,6}$ of the Chow form of the toric variety $\overline{\mathcal{G}_L[\mathbf{u}]} \subset \mathbb{P}^5$ and thus contradict the fact that Chow forms of irreducible subvarieties of \mathbb{P}^5 are irreducible.

Acknowledgement. I want to express special thanks to Alicia Dickenstein for an e-mail saying that some results in [8] reminded her of her paper with Bernd Sturmfels [1]. This made me aware of the interesting results on Chow forms in [1, 4] and triggered the present paper. Chow forms are functions on spaces of *algebraic cycles* in projective space. It is therefore a pleasure to present this work in a volume in honor of Spencer Bloch.

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